MOTIVIC GENERATING SERIES FOR TORIC SURFACE SINGULARITIES

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Abstract. Lejeune-Jalabert and Reguera computed the geometric Poincaré series $P_{geom}(T)$ for toric surface singularities. They raise the question whether this series equals the arithmetic Poincaré series. We prove this equality for a class of toric varieties including the surfaces, and construct a counterexample in the general case. We also compute the motivic Igusa Poincaré series $Q_{geom}(T)$ for toric surface singularities, using the change of variables formula for motivic integrals, thus answering a second question of Lejeune-Jalabert and Reguera's. The series $Q_{geom}(T)$ contains more information than the geometric series, since it determines the multiplicity of the singularity. In some sense, this is the only difference between $Q_{geom}(T)$ and $P_{geom}(T)$.

1. Introduction

Throughout this article, we work over a base field k of characteristic zero, and we denote by k' its algebraic closure.
The classical motivation for the introduction of the motivic generating series can be found in p-adic analysis. Let L be a finite field extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_L$ and uniformizing parameter π. Let X be a variety over $\mathcal{O}_L$.

The Igusa Poincaré series counts approximate solutions modulo π^{n+1}. To be precise, let $\tilde{N}_n$ be the number of points $|X(\mathcal{O}_L/\pi^{n+1})|$, for $n \geq 0$. Then the Igusa Poincaré series is defined to be

$$Q(T) = \sum_{n\geq 0} \tilde{N}_n T^n \ .$$

The Serre-Oesterlé series counts approximate solutions that can be lifted to global solutions on X: putting $\bar{N}_n$ equal to the cardinality of the image of $X(\mathcal{O}_L)$ in $X(\mathcal{O}_L/\pi^{n+1})$, the series is defined as

$$P(T) = \sum_{n\geq 0} \bar{N}_n T^n \ .$$

Both series are known to be rational: Igusa proved the rationality of $Q(T)$ in the hypersurface case, rewriting the series as a p-adic integral and applying resolution of singularities [21]. Denef proved the rationality of $P(T)$ making use of the model-theoretic framework of quantifier elimination and cell decomposition [6].

Since motivic integration is introduced as a formal analogue of p-adic integration, replacing the ring $\mathbb{Z}_p$ by $k[[t]]$ and taking values in the completed localized Grothendieck ring $\hat{\mathcal{M}}_k$, it is natural to translate these series to the motivic setting. This was done by Denef and Loeser (see e.g. [7][12]). Let X be a variety over k.

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The Igusa Poincaré series has a straightforward motivic counterpart. The approximate solutions are given by n-jets, i.e. points on the scheme $\mathcal{L}_n(X)$, which will be introduced in the next section. Instead of counting points, we use the universal additive invariant, mapping a constructible set to its isomorphism class in the Grothendieck ring. In this way, we obtain

$$Q_{geom}(T) = \sum_{n\geq 0}[\mathcal{L}_n(X)]T^n \ .$$

This series is rational in $\mathcal{M}_k[[T]]$, as can be proven by making use of resolution of singularities, and the change of variables formula for motivic integrals. Furthermore, when X is defined over some number field L, this series specializes to the classical Igusa Poincaré series for almost all finite places $\mathcal{P}$. By this we mean the following: we can choose a model over $\mathcal{O}_L$ for X, and count points modulo $\mathcal{P}$, for each finite place $\mathcal{P}$. This operation is denoted by an operator $N_{\mathcal{P}}$; $N_{\mathcal{P}}(X)$ is well-defined for almost all finite places $\mathcal{P}$. Applying $N_{\mathcal{P}}$ termwise to the series Q_{geom} yields the Igusa Poincaré series $Q(T)$ for $X \times \operatorname{Spec} \mathcal{O}_{L_{\mathcal{P}}}$, for almost all finite places $\mathcal{P}$.

A naive generalization for $P(t)$ is obtained by looking at n-jets that can be lifted to arcs on X, that is, by defining the geometric Poincaré series as

$$P_{geom} = \sum_{n\geq 0}[j^n(\mathcal{L}(X))]T^n \ .$$

This series is well-defined, since a theorem of Greenberg guarantees that $j^n(\mathcal{L}(X))$ is constructible, and it is rational in $\mathcal{M}_k[[T]]$ (see [9]). But, in general, this series does not specialize to the Serre-Oesterlé series when X is defined over a number field. The reason for this is that, working scheme-theoretically, we allow extensions of the base field when lifting jets. So instead of counting approximate solutions which can be lifted to a solution in $\mathcal{O}_{L_{\mathcal{P}}}$, we count approximate solutions that can be lifted to a solution in a maximal unramified extension of $\mathcal{O}_{L_{\mathcal{P}}}$, whose residue field is precisely the separable closure of the residue field of $\mathcal{O}_{L_{\mathcal{P}}}$. The arithmetic Poincaré series is designed to remedy this discrepancy. While the Igusa Poincaré series can be computed from a resolution of singularities, the geometric and arithmetic series are very hard to compute. The only known cases so far are formal branches of plane curves [11], toric surfaces [24], and surfaces with an embedded resolution of a simple form [25] (this latter result yields an easy way to recover the formula for toric surfaces).

Let us give an overview of the results in this paper. In Section 2, we recall the definitions of the motivic generating series (Igusa Poincaré series, geometric Poincaré series, arithmetic Poincaré series), with emphasis on the last one. In Section 3, we develop a sufficient condition for the equality of the geometric and arithmetic Poincaré series of toric varieties (Theorem 1), and we show that this condition is always satisfied in the case of a toric surface (Corollary 1). Section 4 contains an example of a toric threefold for which the series differ (Proposition 1). Section 5 gives a concise introduction to the theory of motivic integration. In order to compute the Igusa Poincaré series $Q_{geom}(T)$ of a toric surface, we establish in Section 6 a factorization of the minimal toric resolution into a sequence of blow-ups of smooth subschemes. The actual computation is done in Section 7. In Section 8, we determine which information is contained in the Igusa Poincaré series, by investigating its poles, and we compare it with the formula for the geometric

series P_{geom} in [24]. Theorem 2 shows that $Q_{geom}(T)$ contains more information than P_{geom}, since $Q_{geom}(T)$ also determines the multiplicity of the toric surface singularity. In some sense, this is the only additional information you obtain from $Q_{geom}(T)$.

2. Motivic Poincaré series

Let X be a variety over k, that is, a reduced and separated scheme of finite type over k, not necessarily irreducible. For each positive integer n, the functor from the category of k-algebras to the category of sets, sending an algebra R to the set of $R[[t]]/t^{n+1}R[[t]]$-rational points on X, is representable by a scheme $\mathcal{L}_n(X)$. Since the natural projections $j^n_{n+1} : \mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$ are affine, we can take the projective limit in the category of schemes to obtain the scheme of arcs $\mathcal{L}(X)$. This scheme represents the functor sending a k-algebra R to the set of $R[[t]]$-rational points on X, and comes with natural projections $j^n : \mathcal{L}(X) \to \mathcal{L}_n(X)$, mapping an arc to its n-truncation. We consider $\mathcal{L}(X)$ and $\mathcal{L}_n(X)$ as endowed with their reduced structure. By an arc (resp. n-jet) on X, we always mean a k'-rational point on $\mathcal{L}(X)$ (resp. on $\mathcal{L}_n(X)$), unless explicitly stated otherwise. When X is smooth, the morphisms j^n_{n+1} are Zariski-locally trivial fibrations with fiber $\mathbb{A}^d_k$, where d is the dimension of X.

We now introduce the Grothendieck ring $K_0(Var_k)$ of varieties over k. Start from the free abelian group generated by isomorphism classes $[X]$ of varieties X over k, and consider the quotient by the relations $[X] = [X \setminus X'] + [X']$, where X' is closed in X. A constructible subset of X can be written as a disjoint union of locally closed subsets and determines unambiguously an element of $K_0(Var_k)$. The Cartesian product induces a product on $K_0(Var_k)$, which makes it a ring. We denote the class of the affine line $\mathbb{A}^1_k$ in $K_0(Var_k)$ by $\mathbb{L}$, and the localization of $K_0(Var_k)$ with respect to $\mathbb{L}$ by $\mathcal{M}_k$. On $\mathcal{M}_k$, we consider a decreasing filtration F^m, where F^m is the subgroup generated by elements of the form $[X]\mathbb{L}^{-i}$, with $\dim X - i \leq -m$. We define $\hat{\mathcal{M}}_k$ to be the completion of $\mathcal{M}_k$ with respect to this filtration.

The Grothendieck ring $K_0(Var_k)$ is not very well understood. Recently, Poonen showed that it is not a domain [27]. Bittner proved in [5], using the Weak Factorization Theorem, that $K_0(Var_k)$ can be presented by taking the isomorphism classes of smooth projective varieties as generators, and by considering the relations $[\emptyset] = 0$ and $[Bl_Y X] - [E] = [X] - [Y]$, where X and Y are smooth projective varieties, $Y \subset X$, and E is the exceptional divisor of the blow-up $Bl_Y(X)$ of X along Y. This presentation is important for the construction of additive invariants; it allows one to prove the existence of a ring morphism χ_{mot} from the Grothendieck ring of varieties over k to the Grothendieck ring of Chow motives over k, sending the class of a smooth projective variety to the class of its associated Chow motive, and sending $\mathbb{L}$ to the class of the Tate motive $\mathbb{L}_{mot}$. The existence of this map was proven already in [19]. We denote the image of this morphism by $K_0^{mot}(Var_k)$. A definition of Chow motives can be found in [29]; the idea is that motives should provide some kind of universal cohomology theory. Smooth projective varieties with isomorphic Chow motives have the same cohomology for all known cohomology theories with coefficients in a field of characteristic zero.

The motivic Igusa Poincaré series is defined as

$$Q_{geom}(T) = \sum_{n\geq 0}[\mathcal{L}_n(X)]T^n,$$

while the geometric Poincaré series is by definition

$$P_{geom}(T) = \sum_{n\geq 0}[j^n(\mathcal{L}(X))]T^n \ .$$

The latter series is well defined, since Greenberg's theorem [18] states that we can find a positive integer c such that, for all n, and for each field K containing k, $j^n(\mathcal{L}(X)(K)) = j^n_{nc}(\mathcal{L}_{nc}(X)(K))$. So it follows from Chevalley's theorem [20] that $j^n(\mathcal{L}(X))$ is constructible, and hence determines an element $[j^n(\mathcal{L}(X))]$ in $K_0(Var_k)$. One can define local variants of both series by only considering arcs with origin in some closed subvariety Z of X, as is done in [24]. If we write $\mathcal{L}(X)_Z$ to denote the inverse image $(j^0)^{-1}(Z)$ in $\mathcal{L}(X)$, the local geometric Poincaré series of X at Z is defined as

$$P_{geom,Z}(T) = \sum_{n\geq 0}[j^n(\mathcal{L}(X)_Z)]T^n \ ,$$

and the local Igusa Poincaré series is defined analogously.

As mentioned before, the main characteristic of P_{arith} should be that it behaves well under specialization to $\mathcal{P}$-adic completions of a number field L. The crucial point in the construction is the use of pseudo-finite fields. A pseudo-finite field is an infinite perfect field with exactly one field extension of any given finite degree, and over which every absolutely irreducible variety has a rational point. Their relevance is illustrated by the following theorem of Ax [1]: two ring formulas over $\mathbb{Q}$ are equivalent when interpreted in $\mathbb{F}_p$, for all sufficiently large primes p, if and only if they are equivalent when interpreted in K, for all pseudo-finite fields K containing $\mathbb{Q}$. In this way, they present themselves as natural candidates to control the behaviour of P_{arith} under specialization. Intuitively, they are perfectly suited to detect rationality conditions of the form $(\exists y)y^n = x$, since, for $n > 1$ and p sufficiently large, not every element of $\mathbb{F}_p$ can have an n-th root. This means that the condition $(\exists y)y^n = x$, which is ignored when working over an algebraically closed field, is always brought into account by the much more sensitive pseudo-finite fields. The counterexample in section 4, and the exact definition in [13] of the map χ_c introduced below, will clarify this remark.

A ring formula over a field k is a logical formula φ built from Boolean combinations of polynomial equalities over k, and quantifiers. When allowing extension to an algebraically closed field, we can eliminate quantifiers from φ and thus associate to φ an element of the Grothendieck ring. But these field extensions are exactly what we try to avoid. We will associate to φ an element of $K_0^{mot}(Var_k)\otimes\mathbb{Q}$ in a more subtle way.

Consider the Grothendieck ring $K_0(PFF_k)$ of the theory of pseudo-finite fields containing k. It is generated by classes $[\varphi]$, where φ is a ring formula over k, which are subject to the relations $[\varphi_1 \vee \varphi_2] = [\varphi_1] + [\varphi_2] - [\varphi_1 \wedge \varphi_2]$, whenever φ_1 and φ_2 have the same free variables, and to the relations $[\varphi_1] = [\varphi_2]$, whenever there exists a ring formula ψ over k such that, interpreted over any pseudo-finite field K containing k, ψ defines a bijection between the tuples over K satisfying φ_1 and those satisfying φ_2. Ring multiplication is induced by taking the conjunction of formulas

in disjoint sets of variables. Denef and Loeser [13] constructed a morphism

$$\chi_c : K_0(PFF_k) \to K_0^{mot}(Var_k) \otimes \mathbb{Q} \ .$$

For this construction, it is important to understand the structure of $K_0(PFF_k)$. The theory of quantifier elimination for pseudo-finite fields [15][16], states that quantifiers can be eliminated if one adds some relations to the language, which have a geometric interpretation in terms of Galois covers. This interpretation yields a construction for χ_c. It is important for our purposes that, if our original ring formula φ did not contain any quantifiers in the first place, χ_c maps $[\varphi]$ to the class of the constructible set defined by φ in $K_0^{mot}(Var_k)$.

Now, we are ready to define the arithmetic Poincaré series P_{arith}. We only consider the case where X is a subvariety of some affine space $\mathbb{A}_k^m$; the general case can be dealt with using definable subassignements [13]. It follows from Greenberg's theorem that we can find, for each positive integer n, a ring formula φ_n over k, such that, for all fields K containing k, the K-rational points of $\mathcal{L}_n(X)$ that can be lifted to a K-rational point of $\mathcal{L}(X)$, correspond to the tuples satisfying the interpretation of φ_n in K. We define the arithmetic Poincaré series to be

$$P_{arith} = \sum_{n\geq 0} \chi_c([\varphi_n])T^n \ .$$

The local variant $P_{arith,Z}$, where Z is a closed subvariety of X (defined over k), is defined in the obvious way.

The series is rational over $K_0^{mot}(Var_k)[\mathbb{L}^{-1}] \otimes \mathbb{Q}$. If k is a number field L , we recover the Serre-Oesterlé series for $X \times \operatorname{Spec} \mathcal{O}_{L_\mathcal{P}}$, for almost all finite places $\mathcal{P}$, by applying the operator $N_\mathcal{P}$ to each coefficient of numerator and denominator [13].

3. The arithmetic Poincaré series for toric varieties

Proposition 3.3 in [24] identifies an arc through the zero-dimensional orbit O on an affine toric surface X meeting the embedded torus T with a couple consisting of an arc on T and an N-vector in the interior of the cone σ associated to X. We sketch this identification, which generalizes immediately to arbitrary dimensions. Let X be an affine toric variety, defined over a field k with algebraic closure k', and associated to an n-dimensional cone σ in $N_\mathbb{R} = N \otimes \mathbb{R}$, where N is a lattice of dimension n. Let O be the unique orbit of dimension zero, and let T be the orbit of dimension n. We denote by M the dual lattice of N, and by $\check{\sigma}$ the dual cone of σ in $M_\mathbb{R} = M \otimes \mathbb{R}$.

Let h be an arc on X through O meeting T. This arc can be represented by a coordinate morphism $\psi_h : \check{\sigma} \cap M \to k'[[t]]$. Since h meets T, the image of this morphism does not contain zero, and thus we can define a new mapping $\check{\sigma}\cap M \to \mathbb{N}$ by composing with the function ord_t, measuring the order of a power series in $k'[[t]]$. This mapping extends to a linear form $\nu_h : M \to \mathbb{Z}$, defining a vector in N, which is contained in the interior $Int(\sigma)$ of σ, since $\nu_h(m) > 0$ whenever $m \in \check{\sigma} \cap M$ and $m \neq 0$. If we set $u_h(m) = \psi_h(m)t^{-\nu_h(m)}$ for $m \in \check{\sigma} \cap M$, the mapping u_h extends to a morphism from M to the multiplicative group of units in $k'[[t]]$, which is nothing but an arc on T. Conversely, h can be recovered from ν_h and u_h by setting $\psi_h(m) = t^{<m,\nu_h>}u_h(m)$.

One could say that we have split up the arc h into an order function ν_h and an angular component u_h. In what follows, we identify an arc h with the associated

couple (ν_h, u_h). The smoothness of T reduces the computation of the geometric Poincaré series to a combinatorial analysis of the behaviour of ν_h when h varies.

The next thing we have to do, is to prove that the arcs meeting T suffice to compute the motivic Poincaré series. Let H be the set of arcs through O on X, and let H^* be the subset consisting of arcs meeting T.

Lemma 1 (Moving Lemma)**.** *$j^s(H) = j^s(H^*)$ for each s.*

Proof. Let $h : \tilde{X} \to X$ be a toric resolution of X, corresponding to a subdivision of σ into a simple fan Σ. Let ψ be an arc on X. We will prove that we can deform ψ to an arc ψ' meeting T without changing its s-jet. Let η be the image of the generic point of $\operatorname{Spec} k'[[t]]$ under ψ, and let τ be the face of σ such that η is contained in the orbit O_τ corresponding to τ. Since the choice of a cone of Σ, contained in τ and of the same dimension, yields a section of h over O_τ, we can lift the morphism $\operatorname{Spec} k'((t)) \to X$ induced by ψ to a morphism $\operatorname{Spec} k'((t)) \to \tilde{X}$.

Applying the valuative criterion for properness to the morphism h, we see that the morphism $\operatorname{Spec} k'((t)) \to \tilde{X}$ has a unique extension to an arc $\tilde{\psi}$ on $\tilde{X}$. It is clear that $h(\tilde{\psi}) = \psi$. The variety $\tilde{X}$ being smooth, it is easy to move $\tilde{\psi}$ away from $h^{-1}(X - T)$ (i.e. out of the inverse image of $\mathcal{L}(X - T)$ under h) without changing its s-jet, using a system of local parameters. Now take ψ' to be the image under h of the arc $\tilde{\psi}'$ obtained in this way. □

To prove the equality of the series P_{arith} and P_{geom}, we have to find a convenient way to describe truncations of an arc $h = (\nu_h, u_h)$ in H^*. Let τ be a face of σ, denote by N_τ the sublattice of N generated by $\tau \cap N$, and let M_τ be its dual. Let $\check{G}_\tau$ be a minimal set of generators for the semigroup $\check{\tau} \cap M_\tau$. Suppose that we can find, for each face τ of σ, and for each vector ν_h in $N \cap Int(\tau)$, a basis $\{\mu_i\}_{i=1}^{\dim \tau}$ for M_τ, consisting of elements of $\check{G}_\tau$, such that $< \mu, \nu_h > \geq < \mu_i, \nu_h >$ for each μ in $\check{G}_\tau \setminus \{\mu_i\}_{i=1}^{\dim \tau}$, and for each i such that μ is not contained in the coordinate hyperplane $\lambda_i = 0$ defined by μ_i (*).

Theorem 1. *If (*) holds, then $P_{geom} = P_{arith}$ in $K_0^{mot}(Var_k) \otimes \mathbb{Q}$.*

Proof. Because of the torus action on X, the global series P_{geom} and P_{arith} can be written in terms of the local series P_{geom,x_τ} and P_{arith,x_τ} at the distinguished point x_τ of O_τ, where τ is a face of σ. To be precise,

$$P_{geom} = \sum_{\tau \leq \sigma} (\mathbb{L} - 1)^{n - \dim \tau} P_{geom,x_\tau} \, ,$$

and the analogous statement holds for P_{arith}. Hence, it suffices to prove the theorem for the local series at x_τ. If we denote by Y the complement in X of all orbits $O_{\tau'}$ with τ' a face of σ that is not contained in τ, then Y is isomorphic to the toric variety associated to the cone τ in $N_\mathbb{R}$. Let N' be the sublattice of N generated by $\tau \cap N$, and let Y' be the toric variety associated to the cone τ in $N'_\mathbb{R}$. Since Y is isomorphic to the product of Y' with a torus, it suffices to prove the theorem for the local series at the zero-dimensional orbit O of X.

Let $h = (\nu_h, u_h)$ be an arc in H^*. The angular component u_h is completely determined by $u_h(\mu_i)$, $i = 1, \ldots, n$, where $\{\mu_i\}_{i=1}^n$ is a basis of M, satisfying (*) for $\tau = \sigma$ and for the order vector ν_h. Let s be a positive integer. We define a new angular component u'_h, mapping μ_i to the truncation of $u_h(\mu_i)$ at $t^{s+1-<\mu_i,\nu_h>}$ if $s \geq < \mu_i, \nu_h >$, and to zero in the other case. Because of our supposition, $j^s(\nu_h, u_h)$

equals $j^s(\nu_h, u'_h)$. Moreover, (ν_h, u'_h) has coefficients in the same field as its s-jet, since u'_h can be recovered from the μ_i-coordinates of this jet. So the ring formula φ_s over k, used to define P_{arith}, whose interpretation in any field K containing k is the condition of liftability of a K-rational point of $\mathcal{L}_s(X)$ to a K-rational point of $\mathcal{L}(X)$, is equivalent to the set of equalities and inequalities describing the constructible set $j^s(H)$. By definition of the morphism $\chi_c : K_0(PFF_k) \to K_0^{mot}(Var_k) \otimes \mathbb{Q}$, this implies that the image of $[j^s(H)]$ in $K_0^{mot}(Var_k) \otimes \mathbb{Q}$ equals $\chi_c(\varphi_s)$, which proves the equality of the geometric and the arithmetic Poincaré series. $\square$

An explicit expression for P_{geom} is known only in the case $n = 2$. It follows from the proof that, in order to compute the geometric or arithmetic series for toric varieties of higher dimension, it suffices to consider the local series at the zero-dimensional orbit.

Corollary 1. *The geometric and arithmetic Poincaré series of a toric surface singularity coincide.*

Proof. Let ν_h be a vector in $N \cap Int(\sigma)$, and let $\check{G}$ be the minimal set of generators of the semi-group $\check{\sigma} \cap M$. It is easy to see that there exist elements μ_1, μ_2 in $\check{G}$, forming a $\mathbb{Z}$-basis of M, such that $< \mu, \nu_h > \geq < \mu_i, \nu_h >$ for each μ in $\check{G} \setminus \{\mu_1, \mu_2\}$ and $i = 1, 2$; thus condition (*) is always satisfied in the surface case. $\square$

This basis is used in [24] to calculate the image under the truncation map j^s of the set H^*_ν, consisting of all arcs in H^* with fixed order vector ν.

4. A counterexample

Our supposition (*) is valid when $n = 2$, but does not always hold in higher dimensions.

Proposition 1. *There exists an affine toric threefold V, with zero-dimensional orbit O, such that the local geometric and arithmetic Poincaré series at O differ.*

Proof. Consider the cone $\check{\sigma}$ generated by $(1, 0, 0), (0, 1, 0)$, and $(1, 1, 2)$, and put ν equal to $(2, 2, -1)$. There are three lattice points in $\check{\sigma}$, minimizing ν, and these are precisely the generators - which do not form a basis for M.

The problem that arises is the following: our angular component u_h is determined by its values at the basis $\{(1, 0, 0), (0, 1, 0), (1, 1, 1)\}$. To compute the $(1, 1, 1)$-coordinate of the s-jet of u_h, we only need the first $s - 3$ coefficients of $u_h(1, 1, 1)$; but in order to compute the $(1, 1, 2)$-coordinate, we also need the $(s - 2)$-th. So it may happen that these first $s-3$ coefficients lie in a field k, while the only $(s-2)$-th coefficient yielding the right value for the jet of $u_h(1, 1, 2)$ lies in $k' \setminus k$.

Consider, for instance, the 2-jet mapping $(1, 0, 0)$ and $(0, 1, 0)$ to t^2, $(1, 1, 1)$ to 0, and $(1, 1, 2)$ to $-t^2$. If we define the angular component u by $u(1, 0, 0) = u(0, 1, 0) = 1$ and $u(1, 1, 1) = i$, then (u, ν) lifts this jet to an arc over $\mathbb{C}$. However, the jet is not liftable to a $\mathbb{Q}$-rational point of the arc space, since such an arc (u', ν) has to satisfy $u'(1, 1, 1; 0)^2 = u'(1, 1, 2; 0)u'(1, 0, 0; 0)u'(0, 1, 0; 0) = -1$.

Of course, this does not necessarily mean that this discrepancy actually emerges in the T^2-coefficients of the series; so let us make explicit computations. The set $\{\mu_1 = (1, 0, 0), \mu_2 = (0, 1, 0), \mu_3 = (1, 1, 1), \mu = (1, 1, 2)\}$ is a set of generators for the semigroup $\check{\sigma} \cap M$, and the first three of them form a lattice basis. An arc h, meeting the embedded torus, is, as always, determined by a vector ν in $Int(\sigma) \cap N$,

and an angular component u. What will its 2-jet look like? For each $i = 1 \dots 3$, $u(\mu_i; j)$ can take random values for $0 \leq j \leq 2- < \mu_i, \nu >$. Because (*) is not satisfied, even when we fix these values, we will have some freedom in the choice of the coefficients $u(\mu; j)$, for $0 \leq j \leq 2- < \mu, \nu >$.

If $< \mu, \nu >> 2$, we have no worries, and if $< \mu, \nu >$ is greater than or equal to the maximum of the $< \mu_i, \nu >$, our 2-jet is fixed by the choices we made. So let us suppose it is not. We may as well assume that $< \mu_3, \nu >$ is strictly maximal among the $< \mu_i, \nu >$, because otherwise, ν satisfies (*). If $< \mu, \nu >= 2$, $u_h(\mu, 0)$ is arbitrary, if we allow liftings over an algebraic closure of k. The equalities $< \mu, \nu >= 0$ and $< \mu, \nu >= 1$ cannot occur under the assumptions we made.

This means that the class of $j^2(\mathcal{L}(V)_O)$ in the Grothendieck ring $K_0(Var_k)$, where V is the affine toric threefold with zerodimensional orbit O defined by the dual cone σ of $\check{\sigma}$, and where $\mathcal{L}(V)_O$ denotes the space of arcs on V with origin at O, is equal to

$$\mathbb{L}^9 - \mathbb{L}^6 + 3\mathbb{L}^5 - 6\mathbb{L}^4 + 10\mathbb{L}^3 - 9\mathbb{L}^2 + 3\mathbb{L}\,.$$

As for the arithmetic series, the only jets liftable over the algebraic closure of k, but not necessarily over k itself, are the 2-truncations of arcs (ν, u) with

$$< \mu, \nu >=< \mu_1, \nu >=< \mu_2, \nu >= 2\,,$$

as is the case in our example above, and with $u(\mu_1; 0)u(\mu_2; 0)u(\mu; 0)$ not a square in k. So our ring formula φ_2 becomes

$$\psi \wedge (\exists y) y^2 = x_{\mu_1,2}\, x_{\mu_2,2}\, x_{\mu,2}\,,$$

where ψ is the quantifier-free ring formula over k describing $j^2(\mathcal{L}(V)_O)$, and

$$x_{\mu_i,j},\, x_{\mu,j}, \quad i = 1, 2, 3,\, j = 0, 1, 2$$

are coordinates of 2-jets in the ambient space $\mathbb{A}^4$ associated to our set of generators. Rewriting the formula as a strict disjunction, we see that, in order to prove that the T^2-coefficients in P_{geom} and P_{arith} differ, we have to prove the inequality $\chi_c(\varphi_2') \neq (\mathbb{L} - 1)^3$ in $K_0^{mot}(Var_k) \otimes \mathbb{Q}$, where φ_2' is the ring formula expressing

$$\begin{aligned}(\forall i)(\forall j \neq 2)(x_{\mu_i,j} = 0 \,\wedge\, x_{\mu,j} = 0) \,\wedge\, x_{\mu_3,2} = 0 \,\wedge\, x_{\mu_1,2} \neq 0 \\ \wedge\, x_{\mu_2,2} \neq 0 \,\wedge\, x_{\mu,2} \neq 0 \,\wedge\, (\exists y) y^2 = x_{\mu_1,2}\, x_{\mu_2,2}\, x_{\mu,2}\,,\end{aligned}$$

and where we abuse notation by writing $\mathbb{L}$ for the class of the Tate motive in $K_0^{mot}(Var_k) \otimes \mathbb{Q}$.

Let T be the threefold torus $(\mathbb{A}^1_k \setminus 0)^3$, and consider the Galois cover

$$f : \operatorname{Spec} k[t_i, t_i^{-1}, w]/(w^2 - t_1 t_2 t_3) \to T : (t_1, t_2, t_3, w) \mapsto (t_1, t_2, t_3)$$

with Galois group $\mathbb{Z}_2$. For each field K containing k, the K-rational points of T that lift to a K-rational point of T with respect to f, are exactly the tuples (t_1, t_2, t_3) in K^3 satisfying $(\exists y)(y^2 = t_1 t_2 t_3)$. By definition of the morphism χ_c, this implies that

$$\chi_c(\varphi_2') = \frac{1}{2}(\mathbb{L} - 1)^3\,,$$

which is, of course, exactly what one would expect. Since a threefold torus has non-trivial cohomology, $1/2(\mathbb{L} - 1)^3$ cannot be zero in $K_0^{mot}(Var_k) \otimes \mathbb{Q}$, so $P_{arith} \neq P_{geom}$. $\square$

5. Motivic integration

This section gives a concise survey of some definitions concerning motivic integration. More exact statements and proofs can be found in [9].

Motivic integration is a wonderful theory, which may be considered as an analogue of p-adic integration, replacing the ring $\mathbb{Z}_p$ by $k[[t]]$ and taking values in the completed localized Grothendieck ring $\hat{\mathcal{M}}_k$. Batyrev [3] used p-adic integration and the Weil conjectures to prove that birationally equivalent Calabi-Yau varieties have the same Betti numbers. Kontsevich [23] observed that the development of a geometrical analogue of p-adic integration would allow one to prove stronger results: he used motivic integration to prove that birationally equivalent Calabi-Yau varieties have the same Hodge numbers. Motivic integration was further developed by Batyrev [2][4], and Denef and Loeser [9][10][14].

Let us first introduce a motivic measure μ on the arc space of a variety X of pure dimension d over k. Let A be a subset of $\mathcal{L}(X)$. We call A a cylinder if there exists a positive integer n, and a constructible subset A_n of $\mathcal{L}_n(X)$, such that $A = (j^n)^{-1}(A_n)$. We say A is stable at level n if furthermore, for each $m \geq n$, the projection j^m_{m+1} is a locally trivial fibration over $j^m(A)$, with fiber $\mathbb{A}^d$. In this case, we define $\tilde{\mu}(A)$ to be $[A_n]\mathbb{L}^{-(n+1)d}$ (some authors use $[A_n]\mathbb{L}^{-nd}$ instead). If A is a cylinder, not necessarily stable, we define $\mu(A)$ by cutting out tubular neighbourhoods of $\mathcal{L}(X_{sing})$ in order to obtain stability:

$$\mu(A) = \lim_{e\to\infty} \tilde{\mu}\{A \setminus (j^e)^{-1} j^e(\mathcal{L}(X_{sing}))\} \ .$$

One can check that this limit exists in $\hat{\mathcal{M}}_k$. All these definitions are inspired by the p-adic case. The measure μ is Σ-additive: if a cylinder A can be written as a union of cylinders A_i, $i \in \mathbb{N}$, then $\mu(A) = \sum_i \mu(A_i)$.

We can extend our class of measurable subsets in the following way: consider the norm function $\|.\| : \hat{\mathcal{M}}_k \to \mathbb{R}_{\geq 0}$, mapping an element x to 2^{-n}, where $x \in F^n$ and $x \notin F^{n+1}$. We call a subset A of $\mathcal{L}(X)$ measurable if we can find, for each $\epsilon > 0$, a collection $A_i(\epsilon)$ of cylinders, $i \in \mathbb{N}$, such that the symmetric difference of A and $A_0(\epsilon)$ is contained in $\cup_{i\geq 1} A_i(\epsilon)$, and $\|\mu(A_i(\epsilon)\| \leq \epsilon$ for each $i \geq 1$. One can prove that in this case, $\mu(A) = \lim_{\epsilon\to 0} \mu(A_0(\epsilon))$ is well-defined. So we define the measure of a measurable set by approximating it, using cylinders.

Let $A \subset \mathcal{L}(X)$ be a measurable set, and let α a function from A to $\mathbb{Z} \cup \{\infty\}$. We say $\mathbb{L}^{-\alpha}$ is integrable if $\alpha^{-1}(i)$ is measurable, for each $i \in \mathbb{Z}$, and if the sum $\sum_{i\in\mathbb{Z}} \mu(\alpha^{-1}(i))\mathbb{L}^{-i}$ is well-defined in $\hat{\mathcal{M}}_k$. In this case, this sum is by definition the motivic integral $\int_A \mathbb{L}^{-\alpha} d\mu$.

An important tool in this setting is the change of variables formula [9]. Let X, Y be varieties over k, of pure dimension e, Y smooth, and let $h : Y \to X$ be a proper birational morphism. Let A be a measurable subset of $\mathcal{L}(X)$, and let $\alpha : A \to \mathbb{Z}$ be a function such that $\mathbb{L}^{-\alpha}$ is integrable. Then

$$\int_A \mathbb{L}^{-\alpha} d\mu = \int_{h^{-1}(A)} \mathbb{L}^{-\alpha\circ h - ord_t h^*(\Omega^e_X)} d\mu \ .$$

As can be expected, this formula is often used when h is a resolution of singularities. It allows one to introduce new invariants of X, in terms of a resolution of singularities, which are independent of the chosen resolution, since the definition as a motivic integral on $\mathcal{L}(X)$ is intrinsic. To give an example: using motivic integration, and the change of variables formula, one can prove that the topological zeta

function associated to a regular function f is independent of the chosen resolution for f [8].

Let $V \subset X$ be varieties over k, X smooth of dimension e, and let O be a closed point of V. The embedding of V into the smooth ambient variety X allows us to describe jets on V by means of arcs on X. Let d be a positive integer. We will compute the local Igusa Poincaré series $Q_{geom}(T)$ of V at O by means of the local motivic Igusa zeta function Z, making use of the change of variables formula. By definition,

$$Z(d) = \int_{\mathcal{L}(X)_O} \mathbb{L}^{-ord_t \mathcal{I}^d} d\mu \,,$$

where $\mathcal{I}$ is the defining ideal sheaf of V in X, and we write $\mathcal{L}(X)_O$ to denote the arcs on X with origin in O. Recall that, for an arc ψ in $\mathcal{L}(X)_O$, the order $ord_t \mathcal{I}^d$ is defined as $\min\{ord_t f(\psi) \,|\, f \in \mathcal{I}^d_O,\, f(\psi) \neq 0\}$. Putting $T = \mathbb{L}^{-d}$, we get the classical transformation formula

$$Q_{geom}(T\mathbb{L}^{-e}) = \frac{1 - \mathbb{L}^e Z(T)}{1 - T} \ .$$

If $h : X' \to X$ is any proper birational morphism, with X' smooth,

$$Z(d) = \int_{\mathcal{L}(\tilde{X})_{h^{-1}(O)}} \mathbb{L}^{-ord_t \mathcal{I}^d \circ h - ord_t Jac_h} d\mu$$

where Jac_h is the Jacobian of h. We will take for h an embedded resolution of V in X, because in this case, the behaviour of $ord_t \mathcal{I}^d \circ h$ can be made explicit, allowing us to compute the latter motivic integral.

6. The motivic Igusa Poincaré series

Let V be a singular affine toric surface defined by the cone σ generated by $(1,0)$ and (p,q), where $0 < p < q$ and p,q are relatively prime. Let $(b_1, \ldots, b_s)$ be the entries occurring in the Hirzebruch-Jung continued fraction associated to $q/(q-p)$, and $(c_1, \ldots, c_t)$ the components of the continued fraction of q/p [17][26]. Let furthermore Θ be the union of compact faces of the convex hull of $\sigma \cap N \setminus 0$, and $\check{\Theta}$ be the union of compact faces of the convex hull of $\check{\sigma} \cap M \setminus 0$.

The minimal resolution of V is a toric modification induced by a subdivision of σ into simple cones. The vectors occurring in this subdivision can be listed as follows:

$$v_0 = (1,0),\, v_1 = (1,1), \ldots,\, v_{j+1} = b_j v_j - v_{j-1}, \ldots,\, v_{s+1} = b_s v_s - v_{s-1} = (p,q)\,.$$

The exceptional divisors $E_j \cong \mathbb{P}^1$ of this resolution, $j = 1, \ldots, s$, correspond to the newly introduced vectors v_j, and E_j is known to have self-intersection number $-b_j$.

The c_j have a geometric significance of their own: subdividing $\check{\sigma}$ into simple cones, i.e. taking the minimal set of generators for the semi-group $\check{\sigma} \cap M$, yields an embedding of V into affine $(t+2)$-space; the ideal of V is generated by $x_{i-1}x_{i+1} - x_i^{c_i}$, $i = 1, \ldots, t$.

One can derive the b_i from the c_j, as will be proved using the polar polyhedron Θ^0 associated to Θ. We will describe this connection algorithmically. Read the b_i by order of indexing; a sequence of j 2's induces a dual component $j+3$, unless this sequence contains b_1 or b_s; in that case, the induced number is $j+2$ if only one of both is included, and $j+1$ if the sequence includes both b_1 and b_s. A value $b_i \neq 2$ induces a dual sequence of $b_i - 3$ 2's, unless $i = 1$ or $i = s$; in that case, $b_i - 2$ 2's

appear, or only $b_i - 1$ if $s = 1$. If the successor of $b_i \neq 2$ again differs from 2, these dual 2's must be followed by a 3. Now move on to b_{i+1} and repeat the procedure.

Lemma 2. *Let V be a singular affine toric surface defined by the cone σ generated by $(1,0)$ and (p,q), where $0 < p < q$ and p,q are relatively prime. Let $(b_1,\dots,b_s)$ be the entries occurring in the Hirzebruch-Jung continued fraction associated to $q/(q-p)$, and $(c_1,\dots,c_t)$ the components of the continued fraction of q/p. The algorithm described above computes, with input $(b_1,...,b_s)$, the output string $(c_1,...,c_t)$.*

The algorithm works in both ways, that is, also allows one to deduce the b_i from the c_j.

Proof. Consider the support function $\check{h} : \check{\sigma} \to \mathbb{R}^+$ of Θ, mapping a vector m in $\check{\sigma}$ to

$$\check{h}(m) = \min\{< m,n > \mid n \in \Theta\} \ .$$

We define the polar polyhedron Θ^0 for Θ by

$$\Theta^0 = \{m \in \check{\sigma} \,|\, \check{h}(m) \geq 1\} \ .$$

Let $v_{j(\alpha)}$, $\alpha = 0,\dots,l$, be the vertices of Θ, with $0 = j(0) < \dots < j(l) = s+1$. Let $m_{j(0)}$ be the vector $(0,1)$ in $\check{\sigma}$, and let $m_{j(l+1)}$ be $(q,-p)$. We define $m_{j(\alpha)}$, for $\alpha = 1,\dots,l$, to be the primitive vector in $\check{\sigma} \cap M$ satisfying

$$< m_{j(\alpha)}, v_{j(\alpha-1)} >=< m_{j(\alpha)}, v_{j(\alpha)} >= 1 \ .$$

Now $\check{\Theta}$ is the convex hull of $\{m_{j(\alpha)} \,|\, \alpha = 0,\dots,l+1\} + \check{\sigma}$. For $1 \leq \alpha \leq l-1$, the line segment joining $m_{j(\alpha)}$ and $m_{j(\alpha+1)}$ contains exactly $b_{j(\alpha)} - 1$ lattice points. If we denote by $\{m,m'\}$ the $\mathbb{Z}$-basis for M, dual to $\{v_{j(\alpha)-1}, v_{j(\alpha)}\}$, then $m_{j(\alpha)} = m+m'$, and $m_{j(\alpha+1)} = (b_{j(\alpha)} - 1)m + m'$. This information allows you to derive the algorithm. As an example, let us compute c_1, assuming that $j(1) \neq s$. Since the segment joining $m_{j(1)} = (1,0)$ and $m_{j(2)}$ contains $b_{j(1)} - 1$ lattice points , we find

$$(c_1(b_{j(1)} - 2) - b_{j(1)} + 3, 2 - b_{j(1)})$$

as coordinates of $m_{j(2)}$. On the other hand, the basis dual to $\{v_{j(1)-1}, v_{j(1)}\}$ is $\{(j(1),-1),(1-j(1),1)\}$. Comparing the two expressions for $m_{j(2)}$ yields $c_1 = j(1)+1$. Now $j(1)-1$ is the number j of 2's at the beginning of the series $b_1,\dots,b_s$. One can show that $c_{b_{j(1)}-1} \neq 2$: thus the series $c_1,\dots,c_t$ starts with $b_{j(1)} - 2$ 2's, if $j(1) = 1$, and with the number $j+2$ else. □

In order to compute the Igusa Poincaré series by means of a motivic integral and the change of variables formula, we need to embed the minimal toric resolution for V into an embedded resolution for V in some smooth ambient space. We will factor the canonical toric resolution into a sequence of blow-ups of zero-dimensional orbits, which can be immediately extended to an embedded resolution for V using the embedding in affine space mentioned above. Blowing up the unique zero-dimensional orbit O of V corresponds, by [22], to the toric modification corresponding to the subdivision Σ of σ introducing all primitive vectors normal to the edges of $\check{\Theta}$. Using Θ^0 to describe $\check{\Theta}$, one can show that this comes down to inserting v_1, v_{s-1}, and all v_i determining vertices of Θ, i.e. the v_i for which $b_i \neq 2$. This is a resolution if and only if all b_i with $i \neq 1,s$ are different from 2; in the other case, we have to blow up some more.

The singularities left after blowing up O are all rational double points of type A_c. In fact, they are recovered from the b_i by omitting b_1 and b_s, and isolating all

sequences of 2's in the remaining b_i. Let c be the number of 2's in such a sequence. This number c can be recovered from the c_j: it is equal to $c_j - 3$, with j chosen such that the vertex of $\check{\Theta}$ corresponding to x_j lies on the two edges whose normal directions determine the cone in our fan Σ corresponding to this sequence of 2's. Moreover, each of the c_j which is bigger than 3 will induce a singularity in this way. The singularity will be resolved after blowing up the zero-dimensional orbit corresponding to the associated singular cone (thus inserting 2 vectors, or 1 if $c = 1$) and repeating this procedure $\lfloor c/2 \rfloor$ times.

This factorization allows us to embed our resolution in ambient affine space, simply by blowing up the corresponding points in this space. Let $h : \tilde{X} \to X = \mathbb{A}_k^{t+2}$ be the proper birational morphism obtained in this way, and let $\tilde{V}$ be the strict transform of V (thus $\tilde{V}$ is the canonical resolution surface). The points of $\tilde{V}$ where there's no transversal intersection with the exceptional locus of h correspond to adjacent vectors in the simple subdivision of σ which are introduced in one and the same blow-up. In these points, the intersection multiplicity will be two. This tells us which jets are shared by $\tilde{V}$ and the exceptional locus, preventing us from counting them double.

Let d be a positive integer. We will compute the local Igusa Poincaré series $Q_{geom}(T)$ of V at O by means of the local motivic Igusa zeta function Z, making use of the change of variables formula

$$Z(d) = \int_{\mathcal{L}(X)_O} \mathbb{L}^{-ord_t \mathcal{I}^d} d\mu = \int_{\mathcal{L}(\tilde{X})_{h^{-1}(O)}} \mathbb{L}^{-ord_t \mathcal{I}^d \circ h - ord_t Jac_h} d\mu$$

where $\mathcal{I}$ is the defining ideal sheaf of V in X, and Jac_h is the Jacobian of h. Putting $T = \mathbb{L}^{-d}$, we get that

$$Q_{geom}(T\mathbb{L}^{-(t+2)}) = \frac{1 - \mathbb{L}^{t+2} Z(T)}{1 - T} .$$

Observe that we can recover t from $Q_{geom}(T)$, since the coefficient of the T-term in the series equals $\mathbb{L}^{t+2}$.

7. The computations

Let a be the number of vectors introduced in Σ, i.e. the number of elements in $\{b_2, \ldots, b_{s-1}\}$ differing from 2 augmented by two, and let $b = a - r - 1$ be the number of pairs of adjacent vectors in Σ, that is, pairs of vectors in Σ with multiplicity 1.

We split $\mathbb{L}^{t+2} Z$ up in different terms, corresponding to the classical stratification of the exceptional locus. Let E be the strict transform of the exceptional divisor that is created by blowing up O. The contribution of arcs in $\tilde{X}$ with origin in E, but not in another exceptional divisor or $\tilde{V}$, is clearly equal to

$$Z_1(d) = ([\mathbb{P}^{t+1}] - a[\mathbb{P}^1] + (a-1)) \frac{(\mathbb{L}-1)\mathbb{L}^{-2d-t-2}}{1 - \mathbb{L}^{-2d-t-2}} .$$

Next, we consider arcs with origin in the smooth part of $E' = E \cap \tilde{V}$. In these points, E and $\tilde{V}$ intersect transversally. Let us denote this set of origins by E'^o. Since the order of $\mathcal{I} \circ h$ on E equals 2, in each point the contribution of arcs tangent to neither E nor $\tilde{V}$ amounts to

$$\alpha := (\mathbb{L}^{t+2} - \mathbb{L}^{t+1} - \mathbb{L}^2 + \mathbb{L})\mathbb{L}^{-3d-2t-3} .$$

Counting the arcs tangent to $\tilde{V}$ but not to E yields

$$\beta := (\mathbb{L}-1)(\mathbb{L}^t-1)\frac{\mathbb{L}^{-4d-3t-2}}{1-\mathbb{L}^{-d-t}}\,,$$

while the arcs tangent to E but not to $\tilde{V}$ contribute

$$\gamma := (\mathbb{L}-1)(\mathbb{L}^t-1)\frac{\mathbb{L}^{-5d-3t-4}}{1-\mathbb{L}^{-2d-t-2}}\,.$$

As for the arcs tangent to both E and $\tilde{V}$, we get the same computations at the level of 2-jets, and we obtain

$$\sum_{j=1}^{\infty}(\alpha+\beta+\gamma)\mathbb{L}^{(-3d-2t-2)j}$$

which brings the total contribution of E'^o to

$$Z_2(d) = (a[\mathbb{P}^1]-2(a-1))\frac{(\mathbb{L}^t-1)(\mathbb{L}-1)\mathbb{L}^{-3d-2t-2}}{(1-\mathbb{L}^{-2d-t-2})(1-\mathbb{L}^{-d-t})}.$$

We also have to cope with exceptional divisors emerging during the remainder of the resolution process. The situation is as follows: after blowing up the origin in X, some singularities may remain, situated in the intersection points of exceptional divisors of $\tilde{V}$. They are described by the numbers $d_k = c_k - 3$, indicating the length of the corresponding sequence of 2's in the continued fraction series of $q/(q-p)$. It takes $\lceil d_k/2 \rceil$ blow-ups to resolve them; if d_k is odd, we get a chain of exceptional divisors intersecting $\tilde{V}$ transversally, if d_k is even we get an intersection point of multiplicity 2 in the last stage of the resolution process, and we have to blow up one of the intersection curves to remedy this situation. So for each $c_k > 3$, we get, after simplification, a contribution

$$\begin{aligned} Z^{(k)}(d) \quad = \quad & (\mathbb{L}-1)\{\sum_{j=2}^{\lceil d_k/2 \rceil}\{\,(\mathbb{L}^{t+1}-2\mathbb{L}+1)\frac{\mathbb{L}^{-N'_jd-\nu'_j}}{(1-\mathbb{L}^{-N'_jd-\nu'_j})(1-\mathbb{L}^{-N'_{j+1}d-\nu'_{j+1}})} \\ & +2\frac{(\mathbb{L}-1)(\mathbb{L}^t-1)\mathbb{L}^{-(N'_j+1)d-\nu'_j-t}}{(1-\mathbb{L}^{-d-t})(1-\mathbb{L}^{-N'_jd-\nu'_j})(1-\mathbb{L}^{-N'_{j+1}d-\nu'_{j+1}})}\,\} \\ & +([\mathbb{P}^t]-2)(\mathbb{L}-1)\frac{\mathbb{L}^{-6d-3t-5}}{(1-\mathbb{L}^{-2d-t-2})(1-\mathbb{L}^{-4d-2t-3})} \\ & +2\frac{(\mathbb{L}-1)(\mathbb{L}^t-1)\mathbb{L}^{-7d-4t-5}}{(1-\mathbb{L}^{-d-t})(1-\mathbb{L}^{-2d-t-2})(1-\mathbb{L}^{-4d-2t-3})} \\ & +T^{(k)}\,\}\,, \end{aligned}$$

where $N'_j = 2j$ and $\nu'_j = j(t+1)+1$. The expression for $Z^{(k)} - (\mathbb{L}-1)T^{(k)}$ can be further simplified to $(\mathbb{L}-1)$ times

$$\frac{(\mathbb{L}^{-d-t}+\mathbb{L}^{-d+1}-2\mathbb{L}^{-d}+\mathbb{L}^{t+1}-2\mathbb{L}+1)(\mathbb{L}^{-(\lceil\frac{d_k}{2}\rceil+2)(2d+t+1)-2}+\sum_{j=2}^{\lceil\frac{d_k}{2}\rceil}\mathbb{L}^{-j(2d+t+1)-1})}{(1-\mathbb{L}^{-d-t})(1-\mathbb{L}^{-2d-t-2})(1-\mathbb{L}^{-(\lceil\frac{d_k}{2}\rceil+1)(2d+t+1)-1})}.$$

The term $T^{(k)}$ depends on the parity of d_k. If d_k is odd, then the exceptional divisor created in the final blow-up intersects $\tilde{V}$ transversely along a $\mathbb{P}^1$, so

$$\begin{aligned} T^{(k)} &= ([\mathbb{P}^{t+1}]-[\mathbb{P}^t]-[\mathbb{P}^1]+2)\frac{\mathbb{L}^{-(d_k+3)d-((d_k+1)/2+1)(t+1)-1}}{1-\mathbb{L}^{-(d_k+3)d-((d_k+1)/2+1)(t+1)-1}} \\ &\quad +([\mathbb{P}^1]-2)\frac{(\mathbb{L}^t-1)\mathbb{L}^{-(d_k+4)d-((d_k+1)/2+1)(t+1)-t-1}}{(1-\mathbb{L}^{-(d_k+3)d-((d_k+1)/2+1)(t+1)-1})(1-\mathbb{L}^{-d-t})} . \end{aligned}$$

If d_k is even, then the exceptional divisor of the final blow-up induces two exceptional divisors in $\tilde{V}$, intersecting in a point where the intersection multiplicity of $\tilde{V}$ with the global exceptional divisor is 2. We have to blow up $\tilde{X}$ along one of these divisors to obtain transversal intersection. Hence, $T^{(k)}$ equals

$$\begin{aligned} &([\mathbb{P}^{t+1}]-[\mathbb{P}^t]-2[\mathbb{P}^1]+3)\frac{\mathbb{L}^{-(d_k+2)d-(d_k/2+1)(t+1)-1}}{1-\mathbb{L}^{-(d_k+2)d-(d_k/2+1)(t+1)-1}} \\ &+(2[\mathbb{P}^1]-4)\frac{(\mathbb{L}^t-1)\mathbb{L}^{-(d_k+3)d-(d_k/2+1)(t+1)-t-1}}{(1-\mathbb{L}^{-(d_k+2)d-(d_k/2+1)(t+1)-1})(1-\mathbb{L}^{-d-t})} \\ &+([\mathbb{P}^t]-[\mathbb{P}^{t-1}])\frac{\mathbb{L}^{-(d_k+3)d-(d_k/2+2)(t+1)}}{1-\mathbb{L}^{-(d_k+3)d-(d_k/2+2)(t+1)}} \\ &+\frac{([\mathbb{P}^{t-1}]-1)(\mathbb{L}-1)\mathbb{L}^{-(2d_k+5)d-(d_k+3)(t+1)-1}}{(1-\mathbb{L}^{-(d_k+2)d-(d_k/2+1)(t+1)-1})(1-\mathbb{L}^{-(d_k+3)d-(d_k/2+2)(t+1)})} \\ &+\frac{(\mathbb{L}-1)(\mathbb{L}^t-1)\mathbb{L}^{-(2d_k+6)d-(d_k+4)(t+1)-2}}{(1-\mathbb{L}^{-d-t})(1-\mathbb{L}^{-(d_k+2)d-(d_k/2+1)(t+1)-1})(1-\mathbb{L}^{-(d_k+3)d-(d_k/2+2)(t+1)})} . \end{aligned}$$

To conclude, let us consider arcs with origin in the set of singular points of E': these are intersection points of exceptional divisors of $h|_{\tilde{V}}$ of the first generation. We know that in these points the intersection multiplicity of $\tilde{V}$ and E is 2; the tangent plane of $\tilde{V}$ will be contained in E. We blow up irreducible components of E' in order to remedy this situation. Let F_1 and F_2 be irreducible components of $E \cap \tilde{V}$, intersecting in a point x. We are interested in the contribution

$$\mathbb{L}^{t+2}\int_{\mathcal{L}(\tilde{X})_x}\mathbb{L}^{-ord_t\mathcal{I}^d\circ h-ord_tJac_h}d\mu .$$

Blowing up $\tilde{X}$ along F_2 introduces a $\mathbb{P}^t$-bundle F over F_2. By abuse of notation, we denote the strict transform of the exceptional divisor E again by E, and the strict transform of $\tilde{V}$ by $\tilde{V}$; E, $\tilde{V}$ and F intersect transversally, and E meets the fiber of F over x in a $\mathbb{P}^{t-1}$. This space $\mathbb{P}^{t-1}$ contains the fiber of the intersection of $\tilde{V}$ and F, which is a point. Applying the change of variables formula yields

$$\begin{aligned} Z_3(d) &= ([\mathbb{P}^t]-[\mathbb{P}^{t-1}])(\mathbb{L}-1)\frac{\mathbb{L}^{-3d-2t-2}}{1-\mathbb{L}^{-3d-2t-2}} \\ &\quad +([\mathbb{P}^{t-1}]-1)(\mathbb{L}-1)^2\frac{\mathbb{L}^{-5d-3t-4}}{(1-\mathbb{L}^{-3d-2t-2})(1-\mathbb{L}^{-2d-t-2})} \\ &\quad +(\mathbb{L}-1)^2(\mathbb{L}^t-1)\frac{\mathbb{L}^{-6d-4t-4}}{(1-\mathbb{L}^{-d-t})(1-\mathbb{L}^{-3d-2t-2})(1-\mathbb{L}^{-2d-t-2})} . \end{aligned}$$

Let us summarize these results in a more surveyable way. Let h' be the toric modification which is obtained by taking the canonical resolution, corresponding to the simple subdivision Σ_0 of σ, and blowing up a divisor on $\tilde{V}$ through each

point with intersection multiplicity 2 which results from the resolution of the A_c-singularities; h' does not include the blow-ups of divisors through singular points of E'. Define E_{-1} to be the strict transform of V under h'. Let E_0 be the strict transform of the exceptional divisor that is created in the first blow-up, and let $E_{i,j}$ be the strict transform of the exceptional divisor induced by the j-th blow-up of the singularity corresponding to the i-th sequence of 2's in $b_2, \ldots, b_{s-1}$.

We let I denote the index set

$$\{-1, 0\} \cup \{(i,j) \,|\, i \in \{1, \ldots, r\},\, j \in \{1, \ldots, \lceil (d_i + 1)/2 \rceil\}\}\,.$$

Observe that we allow j to range to $\lceil (d_i+1)/2 \rceil$, because of the extra blow-up we introduced, if necessary, to cope with the point with intersection multiplicity 2. We stratify $\tilde{X}$ in the usual way: for each subset J of I, we define E_J to be $\cap_{\alpha \in J} E_\alpha$, while E_J^o denotes $E_J \setminus \cup_{\alpha \notin J} E_\alpha$.

We attach to each E_α a pair of numerical data (N_α, ν_α) as follows:

$$(N_{-1}, \nu_{-1}) = (1, t),\, (N_0, \nu_0) = (2, t+2), (N_{(i,j)}, \nu_{(i,j)}) = (2(j+1), (j+1)(t+1)+1)\,.$$

If d_i is even and $j = \lceil (d_i+1)/2 \rceil$, we redefine $(N_{(i,j)}, \nu_{(i,j)})$ as

$$(d_i + 3, (d_i/2 + 2)(t+1))\,.$$

Then

$$\begin{aligned} Z(d) \;&=\; \mathbb{L}^{-(t+2)} \sum_{J \subset I, J \not\subseteq \{-1\}} [E_J^o] \prod_{\alpha \in J} \frac{(\mathbb{L}^{codim\, E_\alpha} - 1)\mathbb{L}^{-N_\alpha d - \nu_\alpha}}{1 - \mathbb{L}^{-N_\alpha d - \nu_\alpha}} \\ &\quad + b\, \mathbb{L}^{-(t+2)} \{ Z_3(d) - \frac{(\mathbb{L}^t - 1)(\mathbb{L} - 1)\mathbb{L}^{-3d-2t-2}}{(1 - \mathbb{L}^{-d-t})(1 - \mathbb{L}^{-2d-t-2})} \} \end{aligned}$$

This formula may be considered as a generalization of the formula in terms of an embedded resolution with normal crossings in the hypersurface case. The last term corrects for non-transversal intersection in the singular points of E'.

It follows from results in [9] that this formula holds already over $\mathcal{M}_k$.

8. Extracting information from the motivic zeta function

The question presents itself what information is contained in the Igusa Poincaré series of a toric surface singularity, or, equivalently, in its motivic zeta function.

Theorem 2. *The motivic Igusa Poincaré series $Q_{geom}(T)$ determines the set $\{c_j\}_{j=1}^t$.*

This is the best we can hope for, since the resolution of the singularity O is, intuitively speaking, independent of the order of the c_j, modulo cutting and pasting. Theorems 1 and 2 imply that for toric surface singularities, the motivic Igusa Poincaré series contains more information than the geometric and arithmetic Poincaré series. For instance, Corollary 4.9 in [24] states that $P_{geom}(T)$ is trivial, i.e. equal to $1/(1 - \mathbb{L}^2 T)$, if and only if $s = 1$. In this case, the set $c_1, \ldots, c_t$ will consist entirely of 2's. The geometric series, only considering liftable jets, can't tell you the value of the multiplicity t; the Igusa Poincaré series can, since t appears already in the dimension of the tangent space, i.e. the space of 1-jets. It follows from [24], Corollary 4.8, that this is the only difference between the series $Q_{geom}(T)$ and $P_{geom}(T)$.

Proof of Theorem 2: If $t \neq 1$, we can list the candidate poles of the zeta function $Z(d)$ as follows:

$$-t \leq -\frac{2t+2}{3} \leq -\frac{t+2}{2} < -\frac{(j+1)(t+1)+1}{2(j+1)},\ j \in \{\lceil d_k/2 \rceil \mid c_k > 3\},$$
$$-\frac{(d_k/2+2)(t+1)}{d_k+3},\ d_k \in 2\mathbb{Z}\,.$$

It is important for our purposes that

$$\frac{(d_k/2+2)(t+1)}{d_k+3} > \frac{(d_k/2+1)(t+1)+1}{d_k+2}$$

if $t \neq 1$.

The candidate pole $d = -t$ will always be an actual pole of the motivic zeta function, since in the other case, the denominator $1 - \mathbb{L}^2 T$ would not appear in $Q(T)$; but if we evaluate $(1-\mathbb{L}^2 T)Q(T)$ at $T = \mathbb{L}^{-2}$, we get $[\mathcal{L}_n(X)_O]\mathbb{L}^{-2n}$ as the n-th partial sum, i.e. as the evaluation at $T = \mathbb{L}^{-2}$ of $(1-\mathbb{L}^2 T)Q(T) \bmod T^{n+1}$. It follows from [9], Theorem 7.1, that

$$\lim_{n\to\infty} [j^n \mathcal{L}(X)_O]\mathbb{L}^{-2n} = \mathbb{L}^2 \mu(\mathcal{L}(X)_O)\,,$$

which is non-zero; so the series $[\mathcal{L}_n(X)_O]\mathbb{L}^{-2n}$ diverges, or has a nonzero limit. Hence, we recover t by looking at the smallest pole of Z, which is $-t$ or non-integer - in the latter case, t must be equal to one.

Let us investigate what happens when we specialize to the topological zeta function, as is described in [8]. This yields

$$\begin{aligned} Z_{top}(d) \quad = \quad & \sum_{J\subset I} \chi[E_J^o] \prod_{\alpha\in J} \frac{1}{N_\alpha d + \nu_\alpha} \\ & \qquad + b\,\{Z_{3,\chi}(d) - \frac{1}{(d+t)(2d+t+2)}\}\,, \end{aligned}$$

where $\chi : \mathbb{Z}[\mathbb{L}] \to \mathbb{Z}$ is the topological Euler characteristic; this simply means that we write all coefficients in terms of $\mathbb{L}$, and map $\mathbb{L}$ to 1. The function $Z_{top}(d)$ is well-defined, since it is a specialization of the motivic zeta function, which is defined intrinsically. Poles of Z_{top} will correspond to poles of Z, since the Grothendieck bracket is a finer invariant than the Euler characteristic. Working with Z_{top} instead of Z obviously simplifies the computations, but when the Euler characteristic is too coarse to detect certain poles, or to give useful information about their residues, one is obliged to turn back to Z.

The case $t = 1$ being trivial, we might as well assume that $t > 1$. First suppose $t \neq 2$. The residue of the candidate pole $d = -(2t+2)/3$ is equal to

$$-\frac{b}{3(t-2)^2}\{2t^2 - 5t + 11\}$$

which enables us to recover the value of b. If t happens to be 2, we can still recover b by looking at the residue of the pole $d = -2$, which will have multiplicity 3.

The largest candidate pole of Z_{top} is the one induced by $(N_{i,j}, \nu_{i,j})$, with d_i maximal among the d_k, and $j = \lceil d_i/2 \rceil$. Its residue depends on the numbers δ, ϵ of occurrences of $2j$, resp. $2j-1$, among the d_k. At any rate, it is strictly positive if $r \neq 0$, since it concerns the residue of the largest candidate pole, and all relevant $\chi[E_J^o]$ are positive. Hence, by looking at the largest pole, and its residue with

respect to Z_{top}, we can determine $\lceil d_i/2 \rceil$, and we get a linear relation on δ en ϵ. An additional, independent linear relation is obtained by studying the evaluation of

$$Z(d)(1-\mathbb{L}^{-2(j+1)d-(j+1)(t+1)-1})(1-\mathbb{L}^{-d-t})(1-\mathbb{L}^{-2jd-j(t+1)-1})\\ .(1-\mathbb{L}^{-(2j+3)d-(j+2)(t+1)})$$

at $d=-((j+1)(t+1)+1)/2(j+1)$. Since we are considering the largest candidate pole, one sees without further calculations that the coefficient of $\mathbb{L}^{t+2}$ is equal to $\delta+\epsilon$. Working backwards to $-(2t+3)/4$, we can determine all d_k. $\square$

Remark: Since we don't know wether $\mathcal{M}_k$ is a domain, we should explain what we mean by a pole of a rational function over $\mathcal{M}_k$. An exact definition is given in [28].

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